

# Knizhnik-Zamolodchikov type equations for the root system $B$ and Capelli central elements

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The construction of the well-known Knizhnik-Zamolodchikov equations uses the central element of the second order in the universal enveloping algebra for some Lie algebra. But in the universal enveloping algebra there are central elements of higher orders. It seems desirable to use these elements for the construction of Knizhnik-Zamolodchikov type equations. In the present paper we give a construction of such Knizhnik-Zamolodchikov type equations for the root system  $B$  associated with Capelli central elements in the universal enveloping algebra for the orthogonal algebra.

## 1 Introduction

The Knizhnik-Zamolodchikov equations are a system of differential equations which is satisfied by correlation functions in the WZW theory [1]. Later it turned out that these equations are related with many other areas of mathematics (quantum algebra, isomonodromic deformation).

The Kniznik-Zamolodchikov equations are also interesting as a nontrivial example of an integrable Pfaffian system of Fuchsian type. Mention that the monodromy representation of this system is known explicitly. Thus we get a solution for the Riemann-Hilbert problem in a very particular case.

Let us write the Kniznik-Zamolodchikov equations. They have the form.

$$dy = \lambda \left( \sum_{i \neq j=1}^n \tau_{ij} \frac{d(z_i - z_j)}{z_i - z_j} \right) y,$$

where  $\lambda$  is some complex parameter,  $y(z_1, \dots, z_n)$  is a vector function that takes values in a tensor power  $V^{\otimes n}$  of a representation space  $V$  of a Lie algebra  $\mathfrak{g}$ , and  $\tau_{ij}$  is defined by the formula

$$\tau_{ij} = \sum_s 1 \otimes \dots \otimes \rho(I_s) \otimes \dots \otimes \rho(\omega(I_s)) \otimes \dots 1. \quad (1)$$

Here  $I_s$  is base of a finite-dimensional Lie algebra,  $\{\omega(I_s)\}$  is a base dual to  $\{I_s\}$  with respect to the Killing form. The elements  $I_s$  occur on tensor factors  $i$  and  $j$ , and  $\rho : \mathfrak{g} \rightarrow \text{End}(V)$  is a representation of the Lie algebra  $\mathfrak{g}$ . Thus  $\tau_{ij}$  is a matrix of the operator on  $V^{\otimes n}$ .

The Kniznik-Zamolodchikov equations have singularities on the planes  $z_i = z_j$ . It is natural to look for similar systems that have other singular locus.

Several authors constructed different Kniznik-Zamolodchikov type equations, that have singularities on the reflection hyperplanes of different root system. Thus in [2] a system the Kniznik-Zamolodchikov type equations is constructed that have as the singular set the reflection hyperplanes corresponding to an arbitrary root system. Let us present such a system.

Matsuo considers a root system  $\Delta$ , the set of positive roots is denoted as  $\Delta^+$  and the corresponding Weyl group is denoted as  $W$ . The function  $y$

takes values in the group algebra  $\mathbb{C}[W]$  and the independent variable belong to the dual to the vector space spanned by roots.

Denote a reflection corresponding to the root  $\alpha$  as  $\sigma_\alpha$ .

The Matsuo's Knizhnik-Zamolodchikov type equations contain arbitrary parameters  $\lambda_{|\alpha|}$  depending on the length of the roots  $\alpha$  and one additional parameter  $\lambda$ . The system has the following form.

$$\frac{\partial y}{\partial \xi} = \left( \sum_{\alpha \in \Delta^+} \lambda_{|\alpha|}(\alpha, z) \frac{1}{(\alpha, u)} (\sigma_\alpha - 1) \right) y.$$

Note that the solutions of this system take values in the space  $\mathbb{C}[W]$  and no Lie algebra is involved in its construction explicitly. Nevertheless if one takes in the Matsuo's construction the root system  $A$  and some special representation of the Weyl group one can obtain from Matsuo's system an ordinary Knizhnik-Zamolodchikov system corresponding to the algebra  $\mathfrak{sl}_n$  and its standard representation. Later systems of Matsuo's type were intensively studied by numerous authors (I. Cherednik [3] and others)

There exist other constructions of Knizhnik-Zamolodchikov type equations which are closer to the original equations. One of them is the Leibman's construction of Knizhnik-Zamolodchikov type equations associated with the root system  $B$  [4] and Enriquez's cyclotomic Knizhnik-Zamolodchikov type equations [5].

In [6] we have constructed Knizhnik-Zamolodchikov type equations for the root system  $A$ , but our construction was based on special central elements in the universal enveloping algebra of the orthogonally algebra, namely the Capelli elements. In the present paper we generalize our construction to the root system  $B$  and show that the construction from [6] can be in fact interpreted in some sense as a very special case of the general Leibman's construction.

## 2 Knizhnik-Zamolodchikov equation associated with the root system $B$

Let us explain the Leiman's construction of Knizhnik-Zamolodchikov type equations for the root system  $B$ .

This is a system of type

$$dy = \lambda \left( \sum_{i < j} \frac{d(z_i - z_j)}{z_i - z_j} \tau_{ij} + \sum_{i < j} \frac{d(z_i - z_j)}{z_i - z_j} \mu_{ij} + \sum_i \frac{\nu_i}{z_i} \right) y.$$

Here  $\tau_{ij}$  is defined as in 2

$$\tau_{ij} = \sum_s 1 \otimes \dots \otimes \rho(I_s) \otimes \dots \otimes \rho(\omega(I_s)) \otimes \dots 1, \quad (2)$$

where as before  $I_s$  is a base of a Lie algebra,  $\{\omega(I_s)\}$  is a base dual to  $\{I_s\}$  with respect to the Killing form.

the coefficients  $\mu_{ij}$  and  $\nu_i$  are defined as follows. Denote as  $\sigma$  an involution in the considered Lie algebra  $\mathfrak{g}$ . Then one has

$$\mu_{ij} = \sum_s 1 \otimes \dots \otimes \rho(I_s) \otimes \dots \otimes \rho(\sigma(\omega(I_s))) \otimes \dots 1, \quad (3)$$

the elements  $I_s$  occur on places  $i, j$ ,

$$\nu_i = \frac{1}{2} \sum_s 1 \otimes \dots \otimes \rho(I_s \sigma(\omega(I_s)) + I_s^2) \otimes \dots 1, \quad (4)$$

the elements  $I_s$  occur on the place  $i$ .

Although the Leibman's proof in [4] is done for the case of a simple Lie algebra and is based on calculations in the root base in [5] these is a proof that is valid for an arbitrary finite-dimensional Lie algebra with a fixed central element in  $U(\mathfrak{g})$  of the second order.

### 3 The Lie algebra $\mathcal{T}$ .

In this section we introduce a Lie algebra  $\mathcal{T}$  which plays the crucial role in our construction.

Consider the space of skew-symmetric tensors with  $2n$  indices. Let each index of the skew symmetric tensor take values in the set  $-n, \dots, n$ .

There exist a structure of an associative algebra algebra on this space

$$(e_{a_1} \wedge \dots \wedge e_{a_n} \wedge e_{b_1} \wedge \dots \wedge e_{b_n}) \cdot (e_{-b_1} \wedge \dots \wedge e_{-b_n} \wedge e_{c_1} \wedge \dots \wedge e_{c_n}) := e_{a_1} \wedge \dots \wedge e_{a_n} \wedge e_{c_1} \wedge \dots \wedge e_{c_n}.$$

As a corollary we have a structure of a Lie algebra. Denote this algebra as  $\mathcal{T}$ .

This algebra has a representation on the space of skew-symmetric tensors with  $n$  indices defined by the formula

$$e_{a_1} \wedge \dots \wedge e_{a_n} \wedge e_{b_1} \wedge \dots \wedge e_{b_n} (e_{-b_1} \wedge \dots \wedge e_{-b_n}) := e_{a_1} \wedge \dots \wedge e_{a_n}. \quad (5)$$

The algebra has an involution  $\omega$  defined as follows

$$\omega(e_{a_1} \wedge \dots \wedge e_{a_n} \wedge e_{b_1} \wedge \dots \wedge e_{b_n}) = e_{-a_1} \wedge \dots \wedge e_{-a_n} \wedge e_{-b_1} \wedge \dots \wedge e_{-b_n}.$$

In  $U(\mathcal{T})$  there is a central element of the second order, namely the element

$$C = \sum_{a_1, \dots, a_n, b_1, \dots, b_n} (e_{a_1} \wedge \dots \wedge e_{a_n} \wedge e_{b_1} \wedge \dots \wedge e_{b_n}) \bullet (e_{-a_1} \wedge \dots \wedge e_{-a_n} \wedge e_{-b_1} \wedge \dots \wedge e_{-b_n}),$$

where  $\bullet$  denotes the multiplication in  $U(\mathcal{T})$ . The proof of this fact is essentially contained in [7].

Using general constructions described in Section 5 one can construct Knizhnik-Zamolodchikov type equations associated with the root system  $B$  based with coefficients in the Lie algebra  $\mathcal{T}$ .

In the next section we give an interpretation of these equations as Knizhnik-Zamolodchikov type equations whose construction is based on some certain higher order central elements in  $U(\mathfrak{o}_{2n+1})$ .

## 4 Capelli elements and noncommutative pfaffians

Let us define some certain central elements in the universal enveloping algebra of the orthogonal algebra.

### 4.1 The split realization of the orthogonal algebra

We use the split realization of the orthogonal algebra. This means that we define the orthogonal algebra as the algebra that preserves the quadratic form

$$G = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & & & & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

The row and columns are indexed by  $i, j = -n, -n+1, \dots, n-1, n$ . The zero is skipped in the case  $N = 2n$  and is included in the case  $N = 2n+1$ .

The algebra  $\mathfrak{o}_N$  is generated by matrices

$$F_{ij} = E_{ij} - E_{-j-i}.$$

The commutation relations between these generators are the following

$$[F_{ij}, F_{kl}] = \delta_{kj}F_{il} - \delta_{il}F_{kj} + \delta_{i-k}F_{-jl} + \delta_{-lj}F_{k-i}. \quad (6)$$

## 4.2 Noncommutative pfaffians and Capelli elements

Now let us describe some special higher order central elements in the universal enveloping for the orthogoanl algebra.

Let  $\Phi = (\Phi_{ij})$  be a  $k \times k$  matrix, where  $k$  is even, whose elements belong to some noncommutative ring. The noncommutative pfaffian is defined as follows:

$$Pf\Phi = \frac{1}{2^{\frac{k}{2}}(\frac{k}{2})!} \sum_{\sigma \in S_k} (-1)^\sigma \Phi_{\sigma(1)\sigma(2)} \dots \Phi_{\sigma(k-1)\sigma(k)}.$$

For a subset  $I \subset \{-n, \dots, n\}$  define a submatrix  $F_I = (F_{ij})_{i,j \in I}$ . For this subset put

$$PfF_I := Pf(F_{-i,j})_{-i,j \in I}.$$

**Определение 1.** Put

$$C_k = \sum_{I \subset \{1, \dots, N\}, |I|=k} PfF_I PfF_I,$$

The elements  $C_k$  are the Capelli elements.

**Теорема 1.** [7] For odd  $N$  the elements  $C_k$  are algebraically independent and generate the center, for even  $N$  the same is true if one takes instead the highest Capelli element  $C_N = (PfF)^2$  the central element  $PfF$ .

Below we need two formulas. There proofs can be found in [?].

**Лемма 1.**  $PfF_I = \frac{(\frac{p}{2})!(\frac{q}{2})!}{(\frac{k}{2})!} \sum_{I=I' \sqcup I'', |I'|=p, |I''|=q} (-1)^{(I'I'')} PfF_{I'} PfF_{I''}.$

Here  $(-1)^{(I'I'')}$  is a sign of a permutation of the set  $I = \{i_1, \dots, i_k\}$  that places first the subset  $I' \subset I$  and then the subset  $I'' \subset I$ .

The numbers  $p, q$  are even fixed numbers, they satisfy  $p + q = k = |I|$ .

Let  $\Delta$  be the standard comultiplication in the universal enveloping algebra.

**Лемма 2.**  $\Delta PfF_I = \sum_{I' \sqcup I'' = I} (-1)^{(I'I'')} PfF_{I'} \otimes PfF_{I''}$

Here  $(-1)^{(I'I'')}$  is a sign of a permutation of the set  $I = \{i_1, \dots, i_k\}$  that places first the subset  $I' \subset I$  and then places the subset  $I'' \subset I$ .

### 4.3 The action of Pfaffians on tensors

Let us describe the action of pfaffians in the tensor representations.

**Предложение 1.** On the base vectors  $e_{-2}, e_{-1}, e_0, e_1, e_2$  of the standard representation of  $\mathfrak{o}_5$  the pfaffians  $PfF_I$  where  $|I| = 4$  act as zero operators.

*Proof.* The proposition is proved by direct calculation using the formulae, where  $a \star b = \frac{1}{2}(ab + ba)$

$$PfF_{\widehat{-2}} = F_{0-1} \star F_{-21} - F_{-1-1} \star F_{-20} + F_{-2-1} \star F_{-10}$$

$$PfF_{\widehat{-1}} = F_{0-2} \star F_{-21} - F_{-1-2} \star F_{-20} + F_{-2-2} \star F_{-10}$$

$$PfF_{\widehat{0}} = F_{1-2} \star F_{-21} - F_{-1-2} \star F_{-2-1} + F_{-2-2} \star F_{-1-1}$$

$$PfF_{\widehat{1}} = F_{1-2} \star F_{-20} - F_{0-2} \star F_{-2-1} + F_{-2-2} \star F_{0-1}$$

$$PfF_{\widehat{2}} = F_{1-2} \star F_{-10} - F_{0-2} \star F_{-1-1} + F_{-1-2} \star F_{0-1}$$

□

Prove an analog of the previous statement in an arbitrary dimension



**Предложение 2.** *On the base vectors  $e_{-n}, \dots, e_n$  of the standard representation of  $\mathfrak{o}_N$  the pfaffians  $PfF_I$  for  $|I| > 2$  act as zero operators.*

Put  $q = 4$ ,  $p = k - 4$  in Lemma 1. One has

$$PfF_I e_j = \sum_{I' \sqcup I'' = I, |I'| = k-4, |I''| = 4} \frac{\left(\frac{p}{2}\right)! \left(\frac{q}{2}\right)!}{\left(\frac{k}{2}\right)!} (-1)^{(I' I'')} PfF_{I'} PfF_{I''} e_j.$$

If  $j \notin I''$ , then obviously  $PfF_{I''} e_j = 0$ . If  $j \in I''$ , then using Proposition 1 one also obtains  $PfF_{I''} e_j = 0$ .

Let us find an action of a pfaffian of the order  $k$  on a tensor product of  $< \frac{k}{2}$  vectors, that is on a tensor product  $e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_t}$ , where  $t < k$ .

**Предложение 3.**  *$PfF_I e_{r_2} \otimes e_{r_4} \dots \otimes e_{r_t} = 0$  where  $t < k$*

*Proof.* The following formulae takes place  $\Delta PfF_I = \sum_{I' \sqcup I'' = I} (-1)^{(I' I'')} PfF_{I'} \otimes PfF_{I''}$  (Lemma 2).

By definition one has  $PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k} = (\Delta^k PfF_I) e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k}$ . Since  $t < k$ , the comultiplication  $\Delta^k PfF_I$  contains only summands in which on some place the pfaffian stands whose indexing set  $I$  satisfies  $|I| \geq 4$  (Lemma 2). From Proposition 1 it follows that every such a summand acts as a zero operator.  $\square$

Find an action of a pfaffian of the order  $k$  on a tensor product of  $\frac{k}{2}$  vector, that is on the tensor product  $e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k}$ .

**Предложение 4.** *If  $\{r_2, r_4, \dots, r_k\}$  is not contained in  $I$ , then  $PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k} = 0$ .*

*Otherwise take a permutation  $\gamma$  of  $I$ , such that  $(\gamma(i_1), \gamma(i_2), \dots, \gamma(i_k)) = (r_1, r_2, r_3, \dots, r_{k-1}, r_k)$ . Then*

$$PfF_I e_{r_2} \otimes \dots \otimes e_{r_k} = (-1)^\gamma (-1)^{\frac{k(k-1)}{2}} \sum_{\delta \in \text{Aut}(r_1, r_3, \dots, r_{k-1})} (-1)^\delta e_{-\delta(r_1)} \otimes e_{-\delta(r_3)} \otimes \dots \otimes e_{-\delta(r_{k-3})} \otimes e_{-\delta(r_{k-1})}.$$

*Proof.* By definition one has

$$PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k} = (\Delta^k PfF_I) e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k}.$$

Applying many times the formulae for comultiplication one obtains

$$\Delta^{\frac{k}{2}} PfF_I = \sum_{I^1 \sqcup \dots \sqcup I^k} (-1)^{(I^1 \dots I^k)} PfF_{I^1} \otimes \dots \otimes PfF_{I^k}.$$

Using Proposition 3 one gets that, only the summands for which  $|I^j| = 2, j = 1, \dots, k$  are nonzero operators.

Hence the summation over divisions can be written in the following manner.

$$PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k} = \frac{1}{2^{\frac{k}{2}}} \sum_{\sigma \in S_k} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} \otimes \dots \otimes F_{-\sigma(i_{k-1})\sigma(i_k)} (e_{r_2} \otimes \dots \otimes e_{r_k}) = \frac{1}{2^{\frac{k}{2}}} \sum_{\sigma \in S_k} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} e_{r_2} \otimes \dots \otimes F_{-\sigma(i_{k-1})\sigma(i_k)} e_{r_k}.$$

Consider the expression  $F_{-\sigma(i_1)\sigma(i_2)} e_{r_2}$ . This is  $e_{-\sigma(i_1)}$  if  $\sigma(i_2) = r_2$ , this is  $-e_{-\sigma(i_2)}$  if  $\sigma(i_1) = r_2$  and zero otherwise. Thus the summand is nonzero only if the permutation  $\sigma$  satisfies the following condition. In each pair  $(\sigma(i_{2t-1}), \sigma(i_{2t}))$  either  $\sigma(i_{2t-1}) = r_{2t}$  or  $\sigma(i_{2t}) = r_{2t}$ .

Show that one can consider only the permutations  $\sigma$  such that  $\sigma(i_{2t}) = r_{2t}$ , that is the permutations of type  $(\sigma(i_1), r_2, \sigma(i_2), r_3, \dots, \sigma(i_{k-1}), r_k)$ . But when only summands corresponding to such permutations are considered one must multiply the resulting sum on  $2^{\frac{k}{2}}$ .

It is enough to prove that the permutations  $\sigma = (\sigma(i_1), \sigma(i_2) = r_2, \sigma(i_3), \dots, \sigma(r_k))$  and  $\sigma' = (\sigma(i_2) = r_2, \sigma(i_1), \sigma(i_3), \dots, \sigma(r_k))$  give the same input.

Remind that the input for  $\sigma$  is

$$(-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} e_{r_2} \otimes \dots \otimes F_{-\sigma(i_{k-1})\sigma(i_k)} e_{r_k}.$$

One has from one hand that  $F_{-\sigma(i_1)\sigma(i_2)}e_{r_2} = e_{-\sigma(i_1)}$  and from the other hand  $F_{-\sigma'(i_1)\sigma'(i_2)}e_{r_2} = -e_{-\sigma'(i_2)} = -e_{-\sigma(i_1)}$ . Also one has  $(-1)^\sigma = -(-1)^{\sigma'}$ . Thus the inputs corresponding to  $\sigma$  and  $\sigma'$  are the same.

Hence one can consider the only the permutations  $\sigma$  of type  $(\sigma(i_1), r_2, \sigma(i_2), r_3, \dots, \sigma(i_{k-1}), r_k)$  but multiplying the resulting sum on  $2^{\frac{k}{2}}$ .

For the permutation  $\sigma$  of type  $(\sigma(i_1), r_2, \sigma(i_2), r_3, \dots, \sigma(i_{k-1}), r_k)$  using the definition of  $\gamma$  one gets

$$\begin{aligned} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)}e_{r_2} \otimes \dots \otimes F_{-\sigma(i_{k-1})\sigma(i_k)}e_{r_k} &= (-1)^{(\sigma(i_1)r_2, \dots, \sigma(i_{k-1})r_k)} e_{-\sigma(i_1)} \otimes \\ e_{-\sigma(i_3)} \otimes \dots \otimes e_{-\sigma(i_k)} &= (-1)^{\frac{k(k-1)}{2}} (-1)^\gamma (-1)^\delta e_{-\delta(r_1)} \otimes \dots \otimes e_{-\delta(r_{k-1})}. \end{aligned}$$

Here  $\delta$  is a permutation of the set  $\{r_1, r_3, \dots, r_{k-3}, r_k\}$ .

The equality  $(-1)^{\frac{k(k-1)}{2}} (-1)^\delta (-1)^\gamma = (-1)^\sigma$  was used.

Taking the summation over all permutations  $\delta$ , one gets

$$Pf F_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k} = (-1)^{\frac{k(k-1)}{2}} (-1)^\gamma \sum_{\delta \in \text{Aut}(r_1, \dots, r_{k-1})} (-1)^\delta e_{-\delta(r_1)} \otimes \dots \otimes e_{-\delta(r_{k-1})}.$$

□

Finally from the formula  $Pf F_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_t} = (\Delta^t Pf F_I) e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_t}$ , as in the proof of Proposition 4, one gets the formulae of the action on an arbitrary tensor  $e_{r_2} \otimes \dots \otimes e_{r_t}$ .

**Предложение 5.**  $Pf F_I e_{r_2} \otimes e_{r_4} \dots \otimes e_{r_t} = \sum_{\{j_2, j_4, \dots, j_k\} \subset \{2, 4, \dots, t\}} Pf^{j_2, j_4, \dots, j_t} F_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_t}$ . Here  $Pf^{j_2, j_4, \dots, j_k} F_I$  acts on the tensor multiples with numbers  $j_2, j_4, \dots, j_k$ . It's action is described by Proposition 4

#### 4.4 Pfaffians and representation of the algebra $\mathcal{T}$

Let us give a relation between the representation of the Lie algebra  $\mathcal{T}$  defined by the formula 5 through the action of noncommutative pfaffians.

As a corollary of Proposition 5 we get the following proposition

**Предложение 6.** *In the case  $\mathfrak{o}_{2n+1}$  if  $I \subset J$ ,  $|J| = 2n$  and  $v = e_{i_1} \otimes \dots \otimes e_{i_n}$ , then  $PfF_I PfF_J v = C PfF_J v$ , where the constant  $C$  depend only on  $n$ .*

Now let  $N = 2n + 1$  and  $|I| = 2n$ . Then a subset  $I$  which consists of  $2n$  elements is of type  $\{1, \dots, N\} \setminus i$ . Put

$$PfF_I := PfF_{\hat{i}}.$$

Then  $PfF_{PfF_{I''}J} = 0$  for  $I'' > 2$ .

$$[PfF_{\hat{i}}, PfF_{\hat{j}}] = \frac{1}{n} \sum_{k \neq i, j} (-1)^{(I'I'')} PfF_{\widehat{ikj}} PfF_{\widehat{k}}$$

The sign  $(-1)^{(I'I'')}$  is defined as follows. For an index  $s$  denote as  $\bar{s}$  either  $s$  for  $s < i$ , or  $s - 1$  for  $s > i$ .

Then  $(-1)^{(I'I'')}$  equals  $(-1)^{n-\bar{j}+(n-1)-\bar{k}} = (-1)^{\bar{j}+\bar{k}-1}$  in the case  $j < k$  and  $(-1)^{\bar{j}+\bar{k}}$  in the case  $j > k$ . Denote this sign as  $s_{jk}$ . One has  $s_{jk} = -s_{kj}$ .

Using the theorem 6 one obtains that for a vector  $v$  from a representation of  $\mathfrak{o}_{2n+1}$  with the highest weight  $(1, \dots, 1)$  the following holds.

**Лемма 3.**

$$[PfF_{\hat{i}}, PfF_{\hat{j}}]v = C \sum_{k \neq i, j} s_{jk} PfF_{\widehat{k}}v,$$

where  $C$  is some constant.

The following theorem is proved.

**Теорема 2.** *The representation of the algebra  $\mathcal{T}$  on the space of skew-symmetric tensors with  $n$  indices, given by the formula (5) is given also by the formula*

$$e_{a_1} \wedge \dots \wedge e_{b_n} \mapsto \frac{1}{\sqrt{C}} PfF_{\{a_1, \dots, b_n\}},$$

where the constant is taken from Lemma 3 and the Pfaffian is considered as an operator acting on the space of skew-symmetric tensors with  $n$  indices.

## 5 Knizhnik-Zamolodchikov equations and Capelli elements

Now let us give an interpretation of the Knizhnik-Zamolodchikov type equations associated with the root system  $B$  constructed for the algebra  $\mathcal{T}$  and its representation 5 as a Knizhnik-Zamolodchikov type equations constructed for higher order Capelli central elements.

This fact is an immediate corollary of Theorem 2.

Introduce elements.

$$\tau_{ij} = \sum_{I, |I|=2n} 1 \otimes \dots \otimes \rho(PfF_I) \otimes \dots \otimes \rho(PfF_{-I}) \otimes \dots 1, \quad (7)$$

where the pfaffians occur on places  $i, j$ , and  $\rho$  is a representation of  $\mathfrak{o}_{2n+1}$  on the space of skew-symmetric tensors with  $n$  indices,

$$\mu_{ij} = \sum_s 1 \otimes \dots \otimes \rho(PfF_I) \otimes \dots \otimes \rho(PfF_I) \otimes \dots 1, \quad (8)$$

$$\nu_i = \frac{1}{2} \sum_s 1 \otimes \dots \otimes \rho(PfF_I PfF_{-I} + PfF_I PfF_I) \otimes \dots 1. \quad (9)$$

**Теорема 3.** *The action of elements 2 -4 and 7-9 on the space of skew-symmetric tensors with  $n$  indices coincide.*

As a corollary we get

**Теорема 4.** *The elements 7 -9 satisfy the commutation relation for the coefficients of the Knizhnik-Zamolodchikov type equations associated with the root system  $B$*

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